The Sherrington–Kirkpatrick model near $T = T_c$: expanding around the replica symmetric solution

The Sherrington–Kirkpatrick model near $T = T_c$: expanding around the replica symmetric solution

A Crisanti[1] **and C De Dominicis**[2]

[1] Dipartimento di Fisica, Università di Roma *La Sapienza* and SMC, P.le Aldo Moro 2, I-00185 Roma, Italy
[2] Institut de Physique Théorique, CEA—Saclay—Orme des Merisiers, 91191 Gif sur Yvette, France

E-mail: andrea.crisanti@phys.uniroma1.it and cirano.de dominicis@cea.fr

Abstract

An expansion for the free energy functional of the Sherrington–Kirkpatrick (SK) model, around the replica symmetric (RS) SK solution $Q_{ab}^{(RS)} = \delta_{ab} + q(1 - \delta_{ab})$ is investigated. In particular, when the expansion is truncated to the fourth order in $Q_{ab} - Q_{ab}^{(RS)}$ the full replica symmetry broken (FRSB) solution is explicitly found but it turns out to exist only in the range of temperature $0.549 \ldots \leqslant T \leqslant T_c = 1$, not including $T = 0$. On the other hand, an expansion around the paramagnetic solution $Q_{ab}^{(PM)} = \delta_{ab}$, up to the fourth order, yields a FRSB solution that exists in a limited temperature range $0.915 \ldots \leqslant T \leqslant T_c = 1$.

PACS numbers: 75.10.Nr, 64.70.Pf

1. Introduction

The Sherrington–Kirkpatrick (SK) model is defined by the Hamiltonian [1]

$$\mathcal{H} = -\frac{1}{2} \sum_{i \neq j}^{1,N} J_{ij} \, \sigma_i \sigma_j, \tag{1}$$

where the σ_i are ± 1 Ising spins and the couplings J_{ij} are independent Gaussian random variables of zero mean and variance equal to $1/N$.

The thermodynamic properties of the model are described by the free energy (density) f averaged over the quenched disorder. To overcome the difficulties of averaging a logarithm, the average over the disorder is computed using the so-called *replica trick*

$$-\beta N \overline{f} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}, \tag{2}$$

where $\beta = 1/T$ is the inverse temperature and, as usual, $\overline{(\cdots)}$ denotes the average over the disorder. For n integer Z^n is the partition functions of n identical, non-interacting, replicas of the system. The average over disorder couples the different replicas. Performing this average, and introducing the auxiliary symmetric replica overlap matrix $Q_{ab} = \frac{1}{N} \sum_i \sigma_{ia} \sigma_{ib}$, with $a \neq b$, the disorder-averaged replicated partition functions can be written as [1]

$$\overline{Z^n} = \int \prod_{a<b} \sqrt{\frac{N\beta^2}{2\pi}} \, dQ_{ab} \, e^{N\mathcal{L}[Q]} \tag{3}$$

with the effective Lagrangian (density)

$$\mathcal{L}[Q] = -\frac{\beta^2}{4} \sum_{ab} Q_{ab}^2 + \Omega[Q] - n\frac{\beta^2}{4} \tag{4}$$

$$\Omega[Q] = \ln \mathrm{Tr}_{\sigma_a} \exp\left(\frac{\beta^2}{2} \sum_{ab} Q_{ab} \, \sigma_a \sigma_b \right). \tag{5}$$

The last term in (4) follows from the definition $Q_{aa} = 1$. The normalization factor in (3) gives a sub-leading contributions for $N \to \infty$ and is omitted in the following.

In the thermodynamic limit, $N \to \infty$, the value of the integral in (3) is given by the stationary point value, and the replica-free energy density reads

$$-n\beta f = \mathcal{L}[Q] \tag{6}$$

with Q_{ab} evaluated from the stationary condition

$$\frac{\partial}{\partial Q_{ab}} \mathcal{L}[Q] = 0, \qquad a < b \tag{7}$$

that is from the self-consistent equation

$$Q_{ab} = \frac{\mathrm{Tr}_{\sigma} \, \sigma_a \sigma_b \, \exp\left(\frac{\beta^2}{2} \sum_{ab} Q_{ab} \sigma_a \sigma_b\right)}{\mathrm{Tr}_{\sigma} \exp\left(\frac{\beta^2}{2} \sum_{ab} Q_{ab} \, \sigma_a \sigma_b\right)} = \langle \sigma_a \sigma_b \rangle, \qquad a \neq b. \tag{8}$$

To solve the self-consistent stationary point equation, we have to specify the structure of the matrix Q_{ab}. This is not straightforward since the symmetry of the replicated partition function under replica permutation is broken in the low temperature phase. The replica symmetric (RS) *Ansatz* $Q_{ab} = \delta_{ab} + q(1 - \delta_{ab})$ of Sherrington and Kirkpatrick [1], that assumes the same overlap for any pair of replicas, indeed yields an unphysical negative entropy at zero temperature. Following the parameterization introduced by Parisi [2, 3], the overlap matrix Q_{ab} for R breaking in the replica permutation symmetry is divided into successive boxes of decreasing size p_r, with $p_0 = n$ and $p_{R+1} = 1$, along the diagonal, and the elements Q_{ab} of the overlap matrix are assigned so that

$$Q_{ab} \equiv q_{a \cap b = r} = Q_r, \qquad r = 0, \ldots, R+1, \tag{9}$$

with $1 = Q_{R+1} \geqslant Q_R \geqslant \cdots \geqslant Q_1 > Q_0$. The notation $a \cap b = r$ means that a and b belong to the same box of size p_r but to two *distinct* boxes of size $p_{r+1} < p_r$. The case $R = 0$ gives back the RS solution, while the opposite limit $R \to \infty$ describes a state with an infinite, continuum, number of possible spontaneous breaking of the replica permutation symmetry. It turns out that a physical solution is obtained only in the latter case. Using this structure for Q_{ab}, Parisi and others [2–4] have shown how to obtain solutions with R steps of replica symmetry breaking (RSB) and in particular with $R \to \infty$ (FRSB), and how to construct equations satisfied by $Q(x)$, the continuous limit of the order parameter Q_{ab} for $R \to \infty$ [5].

These equations can be solved in the full low temperature phase [6–11]. However, working directly with $Q(x)$ makes it difficult to keep track, for instance, of the Hessian, and hence of the stability of the solution, since the matrix structure of the overlap matrix Q_{ab} is lost in the continuous limit[3]. The study of the Hessian of the fluctuations around the RSB solution with an arbitrary R from the Lagrangian (4)–(5) is a very hard task. As a result, stability analysis has mostly been investigated near the critical temperature and with the help of a simplified model [12, 13], the so-called *Truncated Model* [2, 14], that similarly to the Landau Lagrangian retains only the main mathematical structure of the expansion of the replicated free energy in powers of Q_{ab} near T_c, where $|Q_{ab}| \ll 1$.

In the present work, we take a different viewpoint and consider the expansion of the Lagrangian (4)–(5) around the replica symmetric *Ansatz* of Sherrington and Kirkpatrick. The main motivation for such an expansion is to obtain a simpler Lagrangian which, while retaining the replica symmetry breaking properties of the original model, is *a priori* valid in the whole low temperature phase. Anticipating our conclusions, we find that the model obtained by truncating the expansion to the fourth order, the minimum order required to have a FRSB solution, while improving the results obtained from the expansions near T_c is valid in a temperature range which does not reach zero temperature.

The outline of the paper is as follows: in section 2, we construct the approximation of $\Omega[Q]$ obtained expanding it around the replica symmetric SK solution $Q_{ab}^{(RS)} = q$ $(a \neq b)$ up to the fourth order in $Q_{ab} - q$. The stationarity equation and its solutions are discussed in section 3. The truncated model was obtained considering the main features of the mathematical structure of the expansion of $\Omega[Q]$ around the paramagnetic solution $Q_{ab}^{(PM)} = 0$ $(a \neq b)$ to the fourth order in Q_{ab}. The parameters entering in the model are, however, usually arbitrary and so it is difficult to make contact with the original SK model. By using the results of section 2 we can determine the coefficients of the expansion and study the properties of the solution. This is done in section 4. Discussion and conclusions are deferred to section 5.

2. Expansion of the free energy functional around the SK solution

To expand the functional $\Omega[Q]$ around the SK solution $Q_{ab} = q$ for $a \neq b$, we consider an overlap matrix Q_{ab} of the form

$$Q_{ab} = \delta_{ab} + q\,(1 - \delta_{ab}) + q_{ab}, \tag{10}$$

where q is given by the SK replica symmetric solution (see below) and q_{ab} the deviation from the replica symmetric solution. Inserting this form of Q_{ab} into the free energy functional (6) yields

$$-n\beta f = n\frac{\beta^2}{4}q^2 - n\frac{\beta^2}{2}q - \frac{\beta^2}{2}q\sum_{ab}q_{ab} - \frac{\beta^2}{4}\sum_{ab}q_{ab}^2$$

$$+ \ln \mathrm{Tr}_\sigma \exp\left[\frac{\beta^2}{2}q\left(\sum_{ab}\sigma_a\right)^2 + \frac{\beta^2}{2}\sum_{ab}^{1,n}q_{ab}\,\sigma_a\sigma_b\right] + \mathrm{O}(n^2). \tag{11}$$

Setting $q_{ab} = 0$, the above expression leads to the Sherrington–Kirkpatrick free energy

$$-\beta f_{\mathrm{SK}} = \frac{\beta^2}{4}q^2 - \frac{\beta^2}{2}q + \overline{\ln\cosh(\beta z)} + \ln 2 + \mathrm{O}(n), \tag{12}$$

---

[3] Paradoxically, it is this continuous limit $R \rightarrow \infty$ that imposes the existence of zero modes (at the bottom of the replicon bands). Indeed, this limit is necessary to transform the replica permutation invariance into a (broken) *continuous* group thus generating Goldstone zero modes.

where the overbar denotes the average over the Gaussian variable z:

$$\overline{g(z)} = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi q}}\, e^{-z^2/2q}\, g(z). \tag{13}$$

Stationarity of f_{SK} with respect to q leads to SK replica symmetric solution

$$q = \overline{\theta^2}, \qquad \theta \equiv \tanh(\beta z). \tag{14}$$

For $q_{ab} \neq 0$, the free energy functional f can be written, expanding the last term in (11) in powers of q_{ab}, as

$$-n\beta f = -n\beta f_{SK} - \frac{\beta^2}{2} q \sum_{ab} q_{ab} - \frac{\beta^2}{4} \sum_{ab} q_{ab}^2$$
$$+ \sum_{k \geq 1} \frac{1}{k!} \left(\frac{\beta^2}{2}\right)^k \left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b\right)^k \right\rangle_c, \tag{15}$$

where the subscript 'c' indicates that only connected contributions, i.e. only those terms that cannot be written as the product of two or more independent sums, must be considered. The angular brackets denote the average

$$\langle g(\boldsymbol{\sigma}) \rangle = \overline{\prod_{a=1}^{n} e^{\beta z \sigma_a} g(\boldsymbol{\sigma})} + O(n). \tag{16}$$

Since $\sigma_a^2 = 1$, the last term in (15) contains only averages of products of spins with different replica index. These are easily evaluated yielding

$$\langle \sigma_{a_1} \cdots \sigma_{a_h} \rangle = \overline{\prod_{a=1}^{n} e^{\beta z \sigma_a} \prod_{l=1}^{h} \sigma_l}$$
$$= \overline{[2\cosh(\beta z)]^{n-h} [2\sinh(\beta z)]^h} + O(n)$$
$$= \overline{\theta^h} + O(n), \qquad a_1 \neq \cdots \neq a_h. \tag{17}$$

Form the study of the truncated model it is known that terms of order $O(q_{ab}^4)$ must be included into the free energy to break the replica symmetry. Thus in the following we shall consider the first four terms of the expansion.

2.1. Term $O(q_{ab})$

The term of order $O(q_{ab})$ is

$$\left\langle \sum_{ab} q_{ab}\, \sigma_a \sigma_b \right\rangle = \sum_{ab} q_{ab} \langle \sigma_a \sigma_b \rangle = \overline{\theta^2} \sum_{ab} q_{ab}. \tag{18}$$

The choice $q = \overline{\theta^2}$, see (14), cancels the linear term in the expansion (15) and removes the tadpoles.

2.2. Terms $O(q_{ab}^2)$

The term of order $O(q_{ab}^2)$ reads

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b\right)^2 \right\rangle = \sum_{\substack{ab \\ cd}} q_{ab}\, q_{cd}\, \langle \sigma_a \sigma_b \sigma_c \sigma_d \rangle. \tag{19}$$

To evaluate this term we have to find all different possible ways of equating the ab indexes to cd indexes with the constraint, imposed by $q_{aa} = 0$, that $a \neq b$ and $c \neq d$. There are three possible cases: all indexes different, a pair of equal indexes and two pairs of equal indexes. By noticing that the spin product averages depend only on the number of different indexes, and not on the value of the indexes, and that the matrix q_{ab} is symmetric, these yield

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b \right)^2 \right\rangle = \overline{\theta^4} \sum_{abcd}{}' q_{ab}\, q_{cd} + 4\,\overline{\theta^2} \sum_{abc}{}' q_{ac}\, q_{cb} + 2 \sum_{ab}{}' q_{ab}^2, \tag{20}$$

since there are four possible ways of equating one index in ab with one index in cd and two was of equating the pair of indexes ab to the pair cd. All sums are restricted to different indexes, this is denoted by the prime '′' over the sum sign. Transforming the restricted sums into unrestricted ones, i.e. sums over free index, one finally ends up with

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b \right)^2 \right\rangle = \overline{\theta^4} \sum_{abcd} q_{ab}\, q_{cd} + 4\,\overline{\theta^2(1 - \theta^2)} \sum_{abc} q_{ac}\, q_{cb} + 2\,\overline{(1 - \theta^2)^2} \sum_{ab} q_{ab}^2. \tag{21}$$

This equation has a simple diagrammatic expression. Indeed denoting q_{ab} by a line and the vertex where two (or more) indexes are equal by a 'dot', the above equation can be written as

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b \right)^2 \right\rangle = \overline{\theta^4} \;\;\rule[0.3em]{2em}{0.4pt}\;\; + 4\,\overline{\theta^2(1 - \theta^2)} \;\;\prec$$

$$+ 2\,\overline{(1 - \theta^2)^2} \;\;\bigcirc\!\!\!-\!\!\bullet \tag{22}$$

More details can be found in appendix B. From this form we can easily see that the first term is a disconnected contribution and hence it does not appear in the free energy (15); therefore, to order $O(q_{ab}^2)$ the free energy reads

$$-n\beta f = -n\beta f_{\mathrm{SK}} + \frac{\beta^4}{4} M \sum_{abc} q_{ac} q_{cb} + \frac{\beta^4}{4} N \sum_{ab} q_{ab}^2 + O(n^2, q_{ab}^3), \tag{23}$$

where

$$M = 2\,\overline{\theta^2(1 - \theta^2)}, \qquad N = \overline{(1 - \theta^2)^2} - T^2. \tag{24}$$

Note that the coefficient N is (minus) the replicon eigenvalue of the replica symmetric solution [15]. The $q_{ab} = 0$ solution is hence unstable below $T = 1$.

2.3. Terms $O(q_{ab}^3)$ and $O(q_{ab}^4)$

These are evaluated as done for the $O(q_{ab}^2)$ by computing all connected contributions that follows from the expansion of the $k = 3$ and $k = 4$ terms in (15). By using a self-explanatory diagrammatic representation these are given by

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b \right)^3 \right\rangle_c = P \;\bigvee\!\!\wedge + Q \;\bigvee\!\!\vert + R \;\bigcirc\!\!-\!\!- + J \;\bigcirc\!\!\!\ominus + K \;\triangle \tag{25}$$

where

$$P = 24\,\overline{\theta^2(1 - \theta^2)^2}, \qquad Q = -16\,\overline{\theta^4(1 - \theta^2)}, \qquad R = -48\,\overline{\theta^2(1 - \theta^2)^2}, \tag{26}$$

$$J = 16\,\overline{\theta^2(1 - \theta^2)^2}, \qquad K = 8\,\overline{(1 - \theta^2)^3} \tag{27}$$

and

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b \right)^4 \right\rangle_c = -A \; \text{⟨diagram⟩} + B \; \text{⟨diagram⟩} - B \; \text{⟨diagram⟩}$$

$$+ C \; \text{⟨diagram⟩} - C \; \text{⟨diagram⟩} + 4D \; \text{⟨diagram⟩}$$

$$- 3D \; \text{⟨diagram⟩} + E \; \text{⟨diagram⟩} - 2E \; \text{⟨diagram⟩}$$

$$+ F \; \text{⟨diagram⟩} + G \; \text{⟨diagram⟩} - H \; \text{⟨diagram⟩} \tag{28}$$

with

$$A = 32\,\overline{\theta^4(1 - 3\theta^2)(1 - \theta^2)}, \qquad B = 384\,\overline{\theta^4(1 - \theta^2)^2}, \qquad C = 384\,\overline{\theta^2(1 - \theta^2)^3}, \tag{29}$$

$$D = 64\,\overline{\theta^2(1 - 3\theta^2)(1 - \theta^2)^2}, \qquad E = 192\,\overline{\theta^2(1 - \theta^2)^2}, \qquad F = 48\,\overline{(1 - \theta^2)^4}, \tag{30}$$

$$G = 32\,\overline{(1 - 3\theta^2)^2(1 - \theta^2)^2}, \qquad H = 96\,\overline{(1 - 3\theta^2)(1 - \theta^2)^3}. \tag{31}$$

Collecting all contributions up to order $\mathrm{O}(q_{ab}^4)$, the replica free energy functional reads

$$-n\beta f = -n\beta f_{\text{SK}} + \frac{1}{4T^4}\left[M \sum_{abc} q_{ac}q_{cb} + N \sum_{ab} q_{ab}^2 \right] + \frac{1}{6(2T^2)^3}\left[P \sum_{abcd} q_{ac}q_{cd}q_{db} \right.$$

$$\left. + Q \sum_{abcd} q_{ad}q_{bd}q_{cd} + R \sum_{abc} q_{ac}^2 q_{cb} + J \sum_{ab} q_{ab}^3 + K \sum_{abc} q_{ac}q_{cb}q_{ba} \right]$$

$$+ \frac{1}{24(2T^2)^4}\left[-A \sum_{abcde} q_{ae}q_{be}q_{ce}q_{de} + B \sum_{abcd} q_{ac}q_{cd}^2 q_{db} \right.$$

$$- B \sum_{abcde} q_{ac}q_{dc}q_{ce}q_{eb} + C \sum_{abc} q_{ac}q_{cb}^2 q_{ba} - C \sum_{abcd} q_{ac}q_{ad}q_{dc}q_{cb}$$

$$+ 4D \sum_{abc} q_{ac}^3 q_{cb} - 3D \sum_{abcd} q_{ab}^2 q_{bc}q_{bd} + E \sum_{abcde} q_{ab}q_{bc}q_{cd}q_{de}$$

$$\left. - 2E \sum_{abcd} q_{ab}q_{bc}q_{cd}^2 + F \sum_{abcd} q_{ab}q_{bc}q_{cd}q_{da} + G \sum_{ab} q_{ab}^4 - H \sum_{abc} q_{ac}^2 q_{cb}^2 \right]$$

$$+ \mathrm{O}(n^2, q_{ab}^5). \tag{32}$$

3. Stationarity equation

The equation for q_{ab} follows from the stationarity condition $(\partial/\partial q_{ab}) f = 0$ applied to the replica free energy functional (32). In the limit $R \to \infty$ this yields

$$\frac{1}{2T^4}[M S_1 + N q(x)] + \frac{1}{6(2T^2)^3}\left[3(P + Q)S_1^2 + R(S_2 + 2S_1 q(x)) + 3J q(x)^2 \right.$$

$$\left. + 6K \left(\int_0^x \mathrm{d}y\, \dot{q}(y)\hat{q}(y) + S_1 q(0) \right) \right] + \frac{1}{24(2T^2)^4}\left[-4A S_1^3 \right.$$

$$+ B\big(2S_1 S_2 - 4S_1^3 + 2S_1^2 q(x)\big) + C\,\Delta(x)$$

$$+ D\big(4S_3 - 6S_1s_2 + 12S_1q(x)^2 - 6S_1^2q(x)\big)$$
$$+ E\big(4S_1^3 - 4S_1S_2 - 4S_1^2q(x)\big)$$
$$+ 12F\left(\int_0^x \mathrm{d}y\, \dot{q}(y)\widehat{q}(y)^2 + S_1^2q(0)\right)$$
$$\left.+ 4G\, q(x)^3 - 4H S_2\, q(x)\right] = 0, \qquad 0 \leqslant x \leqslant x_c, \tag{33}$$

where

$$\Delta(x) = 2\left[\int_0^x \mathrm{d}y\left(\frac{\mathrm{d}}{\mathrm{d}y}q(y)^2\,\widehat{q}(y) + \widehat{q}^2(y)\,\dot{q}(y)\right) + S_1\, q(0)^2 + S_2\, q(0)\right]$$
$$+ (4q(x) - 6S_1)\left[\int_0^x \mathrm{d}y\, \dot{q}(y)\widehat{q}(y) + S_1\, q(0)\right]$$
$$- 3\int_0^x \mathrm{d}y\, q(y)\widehat{q}(y)^2 - 3S_1^2\, q(0) + q(x)^3 \tag{34}$$

and

$$S_n = -\int_0^1 \mathrm{d}x\, q(x)^n = -\int_0^{x_c} \mathrm{d}x\, q(x)^n - (1 - x_c)\, q(x_c)^n. \tag{35}$$

The 'dot' indicates the derivative, $\dot{q}(x) = (\mathrm{d}/\mathrm{d}x)q(x)$, while the 'hat' the replica Fourier transform (RFT), that for $R \to \infty$ reads [17][4]

$$\widehat{q}(x) = \int_x^{x_c} \mathrm{d}y\, y\frac{\mathrm{d}}{\mathrm{d}y}q(y) - q(x_c), \qquad \text{RFT} \tag{36}$$

$$q(x) = -\int_0^x \mathrm{d}y\, \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\widehat{q}(y) + q(0), \qquad \text{inverse RFT}, \tag{37}$$

where $q(0) = q(x = 0)$, and we have neglected the surface term at $x = 1$ since $q(x = 1) = q_{aa} = 0$.

3.1. Solution of the stationarity equation

The complicate integro-differential stationarity equation (33) can be solved reducing it to an ordinary differential equation using the differential operator $\widehat{O} = (1/\dot{q}(x))(\mathrm{d}/\mathrm{d}x)$ to eliminate integrals. Application of $\widehat{O}$ to (33) leads to

$$\frac{N}{2T^2} + \frac{1}{3(2T^2)^3}[RS_1 + 3Jq(x) + 3K\widehat{q}(x)] + \frac{1}{12(2T^2)^4}\left[BS_1^2\right.$$
$$+ C\left(2\int_0^x \mathrm{d}y\, \dot{q}(y)\widehat{q}(y) + \widehat{q}^2(x) + 4q(x)\widehat{q}(x) - 3S_1\,\widehat{q}(x)\right.$$
$$\left.+ 2S_1\, q(0)\right) + D\big(12S_1q(x) - 3S_1^2\big) - 2ES_1^2 + 6F\widehat{q}(x)^2$$
$$\left.+ 6Gq(x)^2 - 2HS_2\right] = 0. \tag{38}$$

The equation is not yet simple enough to be solved. A second application of $\widehat{O}$, and a rearrangement of terms, yields

$$8T^2 X(x) + Y(x)\widehat{q}(x) + U(x)q(x) + Z(x)S_1 = 0, \tag{39}$$

---

[4] The RFT was first introduced, directly in the continuum limit ($R \to \infty$) by Mezard and Parisi [18].

where

$$X(x) = J - Kx, \quad Y(x) = 2C - 4Fx, \quad U(x) = 4G - 2Cx, \quad Z(x) = 4D + Cx. \quad (40)$$

The integral equation (39) can now be transformed into a differential equation dividing it by $Y(x)$ and taking the derivative with respect to x. This leads to the first order differential equation

$$Y(x)[U(x) - Y(x)x]\dot{q}(x) + \mu q(x) + 8T^2\lambda + \nu S_1 = 0, \quad (41)$$

with coefficients

$$\lambda = \dot{X}Y - X\dot{Y} = -2CK + 4FJ \quad (42)$$

$$\mu = \dot{U}Y - U\dot{Y} = -4C^2 + 16FG \quad (43)$$

$$\nu = \dot{Z}Y - Z\dot{Y} = 2C^2 + 16DF. \quad (44)$$

The solution of equation (41) reads

$$q(x) = \Gamma\frac{x - s}{\sqrt{(x - s)^2 + \Delta}} - a - bS_1, \qquad 0 \leqslant x \leqslant x_c, \quad (45)$$

where

$$a = 8T^2\frac{\lambda}{\mu} \qquad b = \frac{\nu}{\mu} \qquad s = \frac{C}{2F} \qquad \Delta = \frac{G}{F} - s^2, \quad (46)$$

and we have absorbed a factor μ into the definition of the integration constant Γ. The quantity S_1 is a function of Γ and x_c (and temperature), see (35). Introducing the auxiliary function

$$h(z) = \int_0^z dx\,\frac{x - s}{\sqrt{(x - s)^2 + \Delta}} + (1 - z)\,\frac{z - s}{\sqrt{(z - s)^2 + \Delta}}$$
$$= \frac{(z - s)(1 - s) + \Delta}{\sqrt{(z - s)^2 + \Delta}} - \sqrt{s^2 + \Delta}, \quad (47)$$

this reads

$$S_1 = \frac{\Gamma\,h(x_c) - a}{b - 1}. \quad (48)$$

The value of Γ, and x_c, is determined from equations (38) and (39). Replacing in equation (39) $q(x)$ with the expression (45) yields a linear equation for Γ. This can be readily solved noticing that since Γ does not depend on x we can just set $x = 0$ and use the identity $\hat{q}(0) = S_1$. This leads to

$$\Gamma = \frac{\Gamma_0}{\Gamma_1 + \Gamma_2\,h(x_c)}, \quad (49)$$

where

$$\Gamma_0 = 4T^2 J(b - 1) + a\,(2G - C - 2D),$$
$$\Gamma_1 = 2G(b - 1)\frac{s}{\sqrt{s^2 + \Delta}}, \quad (50)$$
$$\Gamma_2 = 2Gb - C - 2D.$$

Finally the value of x_c, for a given temperature T, is determined from (38). Again we can take advantage of the fact that x_c does not depend on x and choose in (38) a suitable value for x, e.g. $x = x_c$ or $x = 0$. Setting $x = 0$ into (38) a straightforward algebra leads to the equation

$$2N + \frac{1}{6T^2}[3Jq(0) + (R + 3K)S_1] + \frac{1}{48T^4}\big[6Gq(0)^2 + (6C + 12D)S_1q(0)$$
$$+ (B - 3C - 3D - 3E + 6F)S_1^2 + (C - 2H)S_2\big] = 0, \quad (51)$$
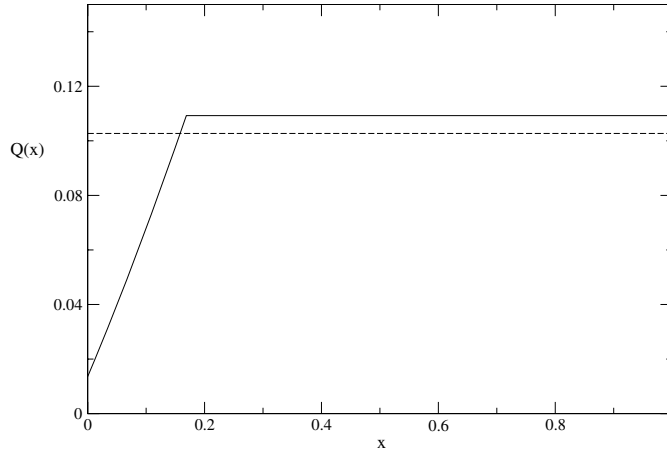
Figure 1. $Q(x)$ versus x at temperature $T = 0.9$. The horizontal dashed line shows the SK solution $Q(x) = q$. For this temperature, we have $x_c = 0.168\,846\ldots$, $Q(x_c) = 0.109\,238\ldots$ $Q(0) = 0.013\,570\ldots$ and $q = 0.102\,701\ldots$.

where

$$S_2 = -\Gamma^2 \left[-(1-x_c)\frac{\Delta}{(x_c-s)^2+\Delta} + I_2(x_c) - I_2(0) + 1 - \frac{b(b-2)}{(b-1)^2}h(x_c)^2 \right]$$
$$+ 2\frac{a}{(b-1)^2}\Gamma h(x_c) - \left(\frac{a}{b-1}\right)^2 \tag{52}$$

and

$$I_2(x) = -\int \mathrm{d}x \, \frac{\Delta}{(x-s)^2+\Delta} = -\sqrt{\Delta}\,\tan^{-1}\left(\frac{x-s}{\sqrt{\Delta}}\right), \qquad \Delta > 0. \tag{53}$$

Solving equation (51) for x_c at fixed T yields the value of $x_c(T)$ that substituted back gives the solution $q(x)$ as a function of temperature. In figures 1 and 2 we show the solutions $Q(x) = q + q(x)$ for two different temperatures.

From the figures one clearly sees that $Q(x = 0) \neq 0$. It grows as the temperature decreases and overcomes q for $T < 0.618\ldots$, see also figure 3. Retaining in the expansion of $\Omega[Q]$ only terms up to order $\mathrm{O}(q_{ab}^4)$ breaks the replica symmetry; however, this approximation is not good enough to change the SK result $Q(x = 0) = q \neq 0$ to the expected one $Q(x = 0) = 0$.[5] To recover the latter one has to add more terms in the expansion, probably all terms.

Below temperature $T = 0.549\ldots$ equation (38) ceases to have a physical solution and only the SK solution $Q(x) = q$ survives. In figure 3 we show the values of $Q(0)$, $Q(x_c)$ and x_c as function of temperature.

3.2. Solution near $T_c = 1$

Near the critical temperature $T_c = 1$, where both q and q_{ab} vanish, the solution of equation (38) can be found as a series expansion in the (small) parameter $\tau = T_c - T$. For example to $\mathrm{O}(\tau^5)$, we have

$$x_c = 2\tau - 4\tau^2 + \frac{40}{3}\tau^3 - \frac{665}{9}\tau^4 + \frac{68\,567}{135}\tau^5 + \mathrm{O}(\tau^6) \tag{54}$$

[5] $Q(x = 0)$ must vanish in the absence of external fields that break the up/down symmetry. For instance in the $q \geqslant 4$ Potts model the symmetry is broken and indeed $Q(0) \neq 0$.
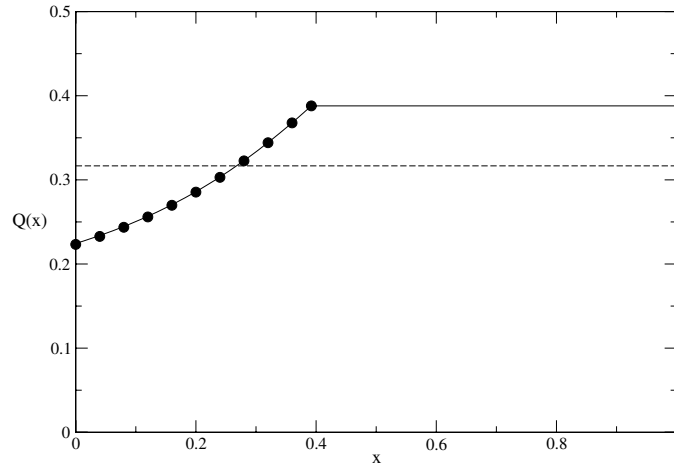
Figure 2. $Q(x)$ versus x at temperature $T = 0.7$. The full line is the result from the expansion around $T = 1$ to order $O(\tau^{13})$, while the circle is obtained from the numerical solution of equation (38). The horizontal dashed line shows the SK solution $Q(x) = q$. For this temperature we have $x_c = 0.3920(6)$, $Q(x_c) = 0.3879(1) \ldots Q(0) = 0.2232(5) \ldots$ and $q = 0.3166(5) \ldots$.
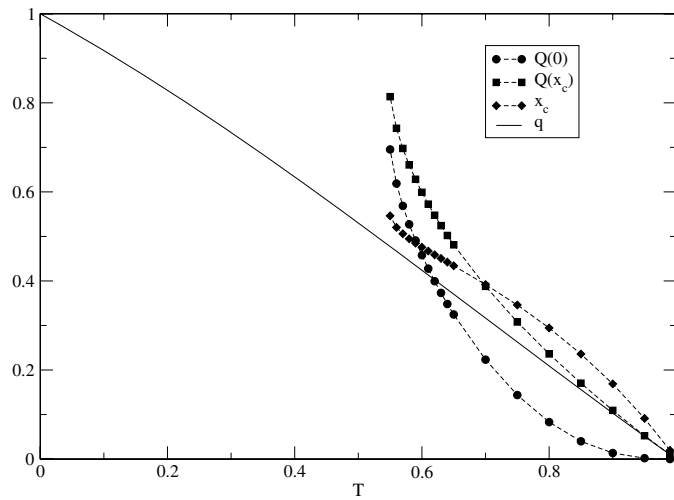
Figure 3. $Q(0)$, $Q(x_c)$ and x_c as function of temperature. The full line is the SK results in $q = \overline{\theta^2}$. The replica symmetry broken solution ends at temperature $T = 0.549 \ldots$.

$$Q(0) = q + q(0) = \frac{56}{3}\tau^3 - \frac{220}{3}\tau^4 + \frac{3968}{9}\tau^5 + O(\tau^6) \tag{55}$$

$$Q(x_c) = q + q(x_c) = \tau + \tau^2 - \tau^3 + \frac{5}{2}\tau^4 - \frac{413}{90}\tau^5 + O(\tau^6). \tag{56}$$

The resulting series are not convergent, but can be handled by using the Padé approximants. We note that the series expansion of x_c has the form of a Stieltjes series $\sum a_n(-\tau)^n$. For these series it is known that the diagonal Padé approximant $P_N^N(\tau)$ gives an upper bound and the

approximant $P_{N+1}^N(\tau)$ a lower bound of the sum [20]. Moreover in the limit of large N both approximants converge, and if they converge to the same limit this is the value of the sum. By using the Padé approximants we were able to use the series expansion almost everywhere in the low temperature phase, where the replica symmetry broken solution exists. For example for temperature $T = 0.7$ by using the series expansion to $O(\tau^{13})$, we have $x_c = 0.3920(6)$, the error being estimated from the difference between the Padé approximants P_N^N and P_{N+1}^N. A comparison between the numerical and the power series solutions is shown in figure 2; there is a rather good agreement.

4. Expansion around the paramagnetic solution

At the critical point $T_c = 1$, the order parameter function $Q(x)$ vanishes, and one can then think of expanding the functional $\Omega[Q]$ around $Q_{ab} = 0$, i.e. the paramagnetic solution. Such an expansion, first considered by Bray and Moore [14], is at the basis of the so-called *Truncated Model* [2] largely used to study the properties of the solution $Q(x)$ near the critical point. See [19] for an extension to more general models. Despite it usefulness, the Truncated Model is a poor approximation for the SK model. Indeed, in the same spirit of the Landau theory of second order transition, it retains only the main mathematical structure of the order $O(Q_{ab}^4)$ expansion of $\Omega[Q]$ around $Q_{ab} = 0$, but with arbitrary coefficients. Using the results of section 2, we can investigate the properties of the $O(Q_{ab}^4)$ approximation of the SK model.

The expansion of the replica free energy functional around $Q_{ab} = 0$ is obtained by setting $q = \theta = 0$ in (32). This yields

$$-n\beta f = n \ln 2 + \frac{1}{4T^4}N \sum_{ab} q_{ab}^2 + \frac{1}{6(2T^2)^3} + K \sum_{abc} q_{ac}q_{cb}q_{ba}$$
$$+ \frac{1}{24(2T^2)^4}\left[F \sum_{abcd} q_{ab}q_{bc}q_{cd}q_{da} + G \sum_{ab} q_{ab}^4 - H \sum_{abc} q_{ac}^2 q_{cb}^2 \right], \qquad (57)$$

with

$$N = 1 - T^2, \qquad K = 8, \qquad F = 48, \qquad G = 32, \qquad H = 96. \qquad (58)$$

Stationarity of (57) with respect to variations of q_{ab} leads to the stationary point equation, that for $R \to \infty$ reads

$$2Nq(x) + \frac{K}{4T^2}\left(\int_0^x dy\, \dot{q}(y)\widehat{q}(y) + S_1\, q(0) \right) + \frac{1}{24T^4}\left[3F\left(\int_0^x dy\, \dot{q}(y)\widehat{q}(y)^2 + S_1^2 q(0) \right) \right.$$
$$\left. + Gq(x)^3 - HS_2 q(x) \right] = 0. \qquad (59)$$

Applying the differential operator $\widehat{O} = (1/\dot{q}(x))(d/dx)$, as done in section 3.1, reduces the above equation to

$$2N + \frac{K}{2T^2}\widehat{q}(x) + \frac{1}{24T^4}[3F\widehat{q}(x)^2 + 3Gq(x)^2 - HS_2] = 0. \qquad (60)$$

This equation is not yet simple enough to be solved. A second application of $\widehat{O}$ leads to

$$-2T^2 Kx - Fx\widehat{q}(x) + Gq(x) = 0. \qquad (61)$$

Dividing this equation by Fx and taking the derivative with respect to x transform the integral equation (61) into a differential equation which, when solved, yields

$$q(x) = \Gamma \frac{x}{\sqrt{x^2 + G/F}}, \qquad 0 \leqslant x \leqslant x_c, \qquad (62)$$

11

with

$$\Gamma = 2T^2 \frac{K}{F} \frac{\sqrt{x_c^2 + G/F}}{x_c + G/F} \tag{63}$$

determined inserting the form (62) of $q(x)$ into (61).

The endpoint x_c is not a free parameter and must be determined as a function of temperature from (60). Inserting $q(x)$ from (62) and

$$\widehat{q}(x) = \frac{G}{F} \frac{q(x)}{x} - 2T^2 \frac{K}{F} \tag{64}$$

into (60), one ends up with the following equation:

$$2N - \frac{K^2}{2F} + \frac{K^2}{6F} \frac{x_c^2 + G/F}{(x_c + G/F)^2} \left[3G + H\left(1 - \sqrt{\frac{G}{F}} \, \tan^{-1}\left(\sqrt{\frac{F}{G}} x_c \right) - \frac{G}{F} \frac{1 - x_c}{x_c^2 + G/F} \right) \right] = 0 \tag{65}$$

that solved for x_c gives the value of $x_c(T)$.

At the critical temperature $T = T_c = 1$, where $N = 0$, x_c vanishes and increases as the temperature is decreased below T_c. Introducing the small parameter $\tau = 1 - T$, the solution of equation (65) can be expressed as a power series. For example to the order $O(\tau^5)$ we have

$$x_c = 2\tau + 12\tau^2 + \tfrac{280}{3}\tau^3 + \tfrac{2437}{3}\tau^4 + \tfrac{37\,641}{5}\tau^5 + O(\tau^6) \tag{66}$$

$$Q(x_c) = \tau + \tau^2 + \tfrac{44}{3}\tau^3 + \tfrac{701}{6}\tau^4 + \tfrac{30\,763}{30}\tau^5 + O(\tau^6) \tag{67}$$

and

$$\Gamma = \frac{1}{\sqrt{6}} - \frac{5}{\sqrt{6}}\tau + \frac{1}{\sqrt{6}}\tau^2 - \frac{23}{\sqrt{6}}\tau^3 - 51\sqrt{\tfrac{3}{2}}\tau^4 - \frac{2972}{5}\sqrt{\tfrac{2}{3}}\tau^5 + O(\tau^6). \tag{68}$$

Note that in this case $Q(x = 0) = 0$.

The maximum allowed value of x_c is 1. Setting $x_c = 1$ into (65), and replacing the constants by their values (58), we find that the replica symmetry broken solution (62) becomes non-physical below the temperature

$$T_{\text{FRSB}} = \tfrac{1}{3}\sqrt{\tfrac{2}{5}\left[21 - \sqrt{6} \, \tan^{-1}\left(\sqrt{\tfrac{3}{2}} \right) \right]} = 0.9148\ldots, \tag{69}$$

where $x_c > 1$. At this temperature $Q(x_c)$ reaches its maximum value

$$\lim_{T \to T_{\text{FRSB}}} Q(x_c) = \tfrac{2}{225}\left[21 - \sqrt{6} \, \tan^{-1}\left(\sqrt{\tfrac{3}{2}} \right) \right] = 0.16737\ldots. \tag{70}$$

We conclude this section noticing that for temperatures above, but close to, T_{FRSB} equation (65) can be solved as power series of $1 - x_c$. We do not report the expansion here.

5. Discussion and conclusions

In this work, we have derived the expansion of the Sherrington–Kirkpatrick model replica free energy functional around the replica symmetric (RS) solution $Q_{ab}^{(\text{RS})} = \delta_{ab} + q(1 - \delta_{ab})$. We have considered in detail the approximation obtained by truncating the expansion to the fourth order in $Q_{ab} - Q_{ab}^{(\text{RS})}$, i.e. the lowest nontrivial approximation to have a continuous replica symmetry breaking. The stationarity equation (33) associated with the approximate free energy functional (32) can be solved and the explicit form of the full replica symmetry broken (FRSB) solution $Q(x)$, for $0 \leqslant x \leqslant x_c$, can be determined. The FRSB solution appears at the critical temperature $T_c = 1$, as the RS solution, and exists only down to the finite temperature $T = 0.549\ldots$. Below this, only the RS solution survives.

A peculiar feature of the FRSB solution is that $Q(x = 0) \neq 0$, and vanishes as $(T_c - T)^3$ as the temperature T approaches the critical temperature $T_c = 1$. This property can be traced back to the fact that the FRSB solution 'opens' around the RS solution $Q^{(RS)}(x)$, i.e., $Q(x = 0) < q < Q(x_c)$, as the temperature decreases below the critical temperature T_c. As the temperature is decreased below T_c the RS solution q increases and drags $Q(x = 0)$ to finite values. We note that at $T = 0.618\ldots$ the value of $Q(x = 0)$ eventually overcomes that of q.

Setting $q = 0$ one recovers the expansion of the replica free energy functional around the paramagnetic solution $Q_{ab}^{(PM)} = \delta_{ab}$ to order $O(Q_{ab}^4)$. This turns out to be rather interesting because such an expansion is at the basis of the truncated model used to study the properties of the FRSB solution $Q(x)$ near the transition. To our knowledge, a study of this approximation with the correct coefficients of the expansion was never done. Indeed the truncated model, and the one in which one keeps all terms generated by the expansion of $\Omega[Q]$ to order $O(Q_{ab}^4)$, has been studied with the arbitrary coefficient. As a consequence of this the existence of a FRSB solution was always taken for granted, but never verified. We have studied the existence of the FRSB solution for this expansion in the last part of this work. Surprisingly it turns out that the FRSB solution exists only close to the critical temperature, in the range of temperature $0.9148\ldots \leqslant T \leqslant 1$. Therefore such expansions truncated to the fourth order cannot be used to study the solution of the SK model near zero temperature.

To summarize,

- the expansion (to the fourth order) around the (replica symmetric) SK solution $Q_{ab}^{(RS)} = \delta_{ab} + q(1 - \delta_{ab})$ yields, in the limit $R \rightarrow \infty$, a $Q(x)$ that does not vanish for x null, in contrast with the exact Parisi solution. The solution exists only in the range $0.549\ldots \leqslant T \leqslant T_c = 1$.
- the expansion (to the fourth order) around the paramagnetic solution $q = 0$ yields a $Q(x)$ that does vanish for x null. But it exists only close to $T_c = 1$, $0.915\ldots \leqslant T \leqslant T_c = 1$.

In this work we have studied the existence of FRSB solutions. Finite RSB solutions may also exist. These, however, may exhibit problems similar to those found for the FRSB solution. For example, inserting the replica symmetric *Ansatz* $q_{ab} = q(1 - \delta_{ab})$ into the free energy functional (57), and taking the limit $n \rightarrow 0$, or expanding the SK free energy (12) around $q = 0$ to the fourth order in q, one ends up with

$$-\beta f = \ln 2 - \frac{1 - T^2}{4T^4} q^2 + \frac{1}{3T^6} q^3 - \frac{17}{24T^8} q^4. \tag{71}$$

Stationarity with respect to variations of q yields the paramagnetic solution $q = 0$ and the RS solution

$$q = \frac{T^2}{17}[3 - \sqrt{3(17T^2 - 14)}]. \tag{72}$$

The latter correctly vanishes at the critical point $T = T_c = 1$, but exists only down to temperature $T = \sqrt{14/17} \simeq 0.907\ldots$, slightly below the lower end $T = 0.915\ldots$ of the FRSB solutions, where the quantity under the square root becomes negative.

The situation is only slightly better considering the expansion around the replica symmetric SK solution since now the RS solution exists down to $T = 0$. The RS *Ansatz* $q_{ab} = \delta q(1 - \delta_{ab})$ yields indeed, besides the trivial solution $\delta q = 0$, a $\delta q \neq 0$ solution leading to the unphysical result $Q = q + \delta q \simeq -\frac{31}{3}(1 - T)^3$ as $T \sim 1^-$. The reason is that the expansion around the SK solution includes the contribution of more diagrams: all diagrams that are needed to build the SK free energy. In this sense this expansion is a better approximation for the SK model, as also reflected by the larger temperature range where the FRSB solution exist.

The improvement is only apparent since for both expansions the RS solution, as well as the paramagnetic solution for $T < 1$, has a negative replicon mass and is, hence, unstable.

For what concerns the FRSB solution, whether in the full expansion (57) or in the truncated model, they both have the same stability properties in the most dangerous sector, i.e. in the replicon subspace (by virtue of the Ward–Takahashi identities [16]). Thus the FRSB, where it does exist, is marginally stable with null replicon masses. We believe that this feature remains true for the expansion around the SK solution as well.

Despite this, the limited range of temperature (and not including $T = 0$) in which these expansions to the fourth order exist, makes them of little help to study the properties of the SK model near zero temperature. To extend the range of validity one should retain more terms in the expansion, and probably all terms (or infinite subseries thereof) since the particular structures of the expansion (in powers of β) may otherwise lead to difficulties for very low temperatures. We observe that to overcome this problem a construction, based upon an expansion around a spherical approximation, which leads instead to an expansion in T, has been recently proposed [21].

Appendix A. The truncated model

The truncated model is defined by the free energy

$$-nf = \frac{\tau}{2} \sum_{ab} q_{ab}^2 + \frac{w}{6} \sum_{abc} q_{ac} q_{cb} q_{ba} + \frac{u}{12} \sum_{ab} q_{ab}^4, \tag{A.1}$$

with $\tau = 1 - T$ and w and u arbitrary and positive. Comparison of (A.1) and (57) shows that

$$\tau = \frac{N}{2T^3}, \qquad w = \frac{K}{8T^5}, \qquad u = \frac{G}{32T^7}, \tag{A.2}$$

while $F = H = 0$. We can then read the equation for $q(x)$ directly from section 4. Setting $F = 0$ in (61) one readily obtains the known linear form of $q(x)$ for the truncated model

$$q(x) = 2T^2 \frac{K}{G} x = \frac{w}{2u} x, \qquad 0 \leqslant x_c \leqslant x_c. \tag{A.3}$$

Finally setting $F = H = 0$ in (60), and using the above linear form of $q(x)$, yields

$$2N - \frac{K^2}{G} x_c + \frac{K^2}{2G} x_c^2 = 0 \tag{A.4}$$

that gives x_c as a function of temperature. The value of x_c is zero for $T = 1$ and increases as T decreases below 1. By setting $x_c = 1$ into (A.4) leads to the critical temperature

$$T_{\text{trm}} = \sqrt{1 - \frac{K^2}{4G}} \tag{A.5}$$

below which the replica symmetry broken solution ceases to exist. If $1 - K^2/4G < 0$ the solution exists down to $T = 0$. If we use the values $K = 8$ and $G = 32$ we have $T_{\text{trm}} = 1/\sqrt{2} = 0.707\ldots$.

We note that due to the presence of T-factors in the relation between (N, K, G) and (τ, w, u), the critical temperature has a slightly different form if expressed in the latter:

$$T_{\text{trm}} = 1 - \frac{w^2}{4u} \tag{A.6}$$

and is valid if w and u are temperature independent.

14

Appendix B. Terms O(q_{ab}^2): details

The terms of order O(q_{ab}^2) are given by

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a\sigma_b \right)^2 \right\rangle = \sum_{\substack{ab \\ cd}} q_{ab}\, q_{cd}\, \langle \sigma_a\sigma_b\sigma_c\sigma_d \rangle. \tag{B.1}$$

To evaluate this term we have to find all possible ways of equating the ab indexes to cd indexes, with the constraint $a \neq b$ and $c \neq d$ since $q_{aa} = 0$. In the following to denote that a group of indexes must be all different we shall write them in parenthesis; hence, in the present case we have to write (ab) and (cd).

We clearly have three possible cases: all different, one equal, two equals. For later use it is useful to represent them graphically. If we denote q_{ab} by a straight line then the case of all different indexes is represented as

$$\tag{B.2}$$

and the value of the average is $\overline{\theta^4}$ since all spin indexes are different. Next there are four possible ways of equating one (ab) index to one (cd) index. These are

$$\tag{B.3}$$

where the indexes connected by a dashed line are equal. In this case two indexes in the average are equal, so only two spins survive and the spin average gives $\overline{\theta^2}$.

Finally there are two possible ways of equating indexes (ab) and indexes (cd) with the constraint $a \neq b$ and $c \neq d$ and reads

$$\tag{B.4}$$

In this case we have two pairs of equal indexes in the average, so all spins disappear and the average gives 1.

To evaluate (B.1) we have to sum each diagram over a, b, c, d; then it is easy to realize that since the matrix q_{ab} is symmetric all four diagrams in (B.3) give the same contribution, and so do the two diagrams in (B.4). These will be denoted as

$$\tag{B.5}$$

respectively.

Collecting all terms we have

$$\left\langle \left(\sum_{ab} q_{ab}\, \sigma_a \sigma_b \right)^2 \right\rangle = \overline{\theta^4} \sum_{(abcd)} q_{ab}\, q_{cd} + 4\,\overline{\theta^2} \sum_{(abc)} q_{ac}\, q_{cb} + 2 \sum_{(ab)} q_{ab}^2. \tag{B.6}$$

The restricted sums can be transformed into unrestricted sums by inserting a factor $(1 - \delta_{ab})$ for each pair of indexes (ab) to enforce the constraint and removing the constraint over the indexes. By expanding now the resulting products of $(1 - \delta)$'s, each restricted sum is finally expressed as a combination of unrestricted sums. Diagrammatically we have

and

Inserting these expressions into (B.6) after simple manipulations we end up with

The first term is disconnected and hence it does not contribute to the free energy; therefore, to order $\mathrm{O}(q_{ab}^2)$ the free energy reads

$$-n\beta f = -n\beta f_{\mathrm{SK}} - \frac{\beta^2}{4}[1 - \beta^2 \overline{(1 - \theta^2)^2}] \sum_{ab} q_{ab}^2 + \frac{\beta^4}{2} \overline{\theta^2(1 - \theta^2)} \sum_{abc} q_{ac} q_{cb} + \mathrm{O}(q_{ab}^3). \tag{B.10}$$

References

[1] Sherrington D and Kirkpatrick S 1978 *Phys. Rev.* B **17** 4384
[2] Parisi G 1979 *Phys. Rev. Lett.* **43** 1754
[3] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** 1101
[4] Duplantier B 1981 *J. Phys. A: Math. Gen.* **14** 283
[5] Parisi G 1983 *Phys. Rev. Lett.* **50** 1946
[6] Crisanti A and Rizzo T 2002 *Phys. Rev.* E **65** 46137
[7] Crisanti A, Rizzo T and Temesvari T 2003 *Eur. Phys. J.* B **33** 203
[8] Pankov S 2006 *Phys. Rev. Lett.* **96** 197204
[9] Oppermann R and Sherrington D 2005 *Phys. Rev. Lett.* **95** 197203
[10] Oppermann R, Schmidt M J and Sherrington D 2007 *Phys. Rev. Lett.* **98** 127201
[11] Schmidt M J and Oppermann R 2008 *Phys. Rev.* E **77** 061104
[12] Bray A J and Moore M A 1978 *Phys. Rev. Lett.* **41** 1068
[13] De Dominicis C and Kondor I 1983 *Phys. Rev.* B **27** 606
[14] Bray A and Moore M 1979 *J. Phys. C: Solid State Phys.* **12** 79
[15] de Almeida J R and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
[16] De Dominicis C, Temesvari T and Kondor I 1998 *J. Phys. IV France* **8** 13 (arXiv:cont-mat/9802166) (The equation numbering contains errors; the reader should rather consult the cond-mat version.)
[17] De Dominicis C, Carlucci D M and Temesvari T 1997 *J. Phys. I France* **7** 105
[18] Mezard M and Parisi G 1991 *J. Phys. I France* **1** 809
[19] Schelkacheva T I, Tateyeva E E and Chtchelkatchev N M 2006 *Phys. Lett.* A **358** 222
[20] Bender C and Orszag S A 1999 *Advanced Mathematical Methods For Scientists and Engineers* (Berlin: Springer)
[21] Crisanti A, De Dominicis C and Sarlat T 2009 *Eur. Phys. J.* B at press (arXiv:cond-mat/0909.2556)